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ON THE EXPANSION OF CONFLUENT HYPERGEOMETRIC FUNCTIONS IN TERMS OF BESSEL FUNCTIONS

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On the expansion of confluent hypergeometric functions in terms of Bessel functions *)

by

N.M. Temme

ABSTRACT

For the confluent hypergeometric functions U(a,b,z) and M(a,b,z) asymptotic expansions are given for $a \to \infty$. The expansions contain modified Bessel functions. For real values of the parameters rigorous error bounds are given.

KEY WORDS & PHRASES: *confluent hypergeometric functions, Kummer's functions, asymptotic expansions, modified Bessel functions*

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

The confluent hypergeometric functions M(a,b,z) and U(a,b,z) are solutions of Kummer's differential equation

$$(1.1) \qquad zy'' + (b-z)y' - ay = 0.$$

If $a \neq 0, -1, -2, \ldots$, M and U are linearly independent. In general U is singular at z = 0, whereas M is an entire function with the expansion

$$(1.2) \qquad M(a,b,z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b)z^n}{\Gamma(a)\Gamma(b+n)n!} .$$

A convenient explicit representation for U is the integral

$$(1.3) \qquad U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^{\infty} t^{a-1}(1+t)^{b-a-1} e^{-zt}\, dt,$$

where Re a > 0, Re z > 0, $b \in \mathbb{C}$. A well-known integral for M is

$$(1.4) \qquad M(a,b,z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^{1} t^{a-1}(1-t)^{b-a-1} e^{zt}\, dt,$$

with Re b > Re a > 0. This restriction on a and b is not suitable for the present investigations. From SLATER (1960) we take the variant

$$(1.5) \qquad M(a,b,z) = \frac{\Gamma(1+a-b)\Gamma(b)}{\Gamma(a)2\pi i} \int_0^{(1^+)} t^{a-1}(t-1)^{b-a-1} e^{zt}\, dt$$

which is valid for Re a > 0 and $1 + a - b \neq 0, -1, -2, \ldots$. The contour of integration in (1.5) is shown in Figure 1; of course it can be deformed by using Cauchy's theorem for integrals of analytic functions of a complex variable. The many-valued functions in the integrand of (1.5) are supposed to be real for t > 1. Considered as a function of b, M has simple poles at b = 0, -1, -2, -3, ... This is described by the factor Γ(b) in (1.2), (1.4) and (1.5). If b and a are both negative integers such that $b \leq a$ then M(a,b,z) is not defined. A useful reflection formula with respect to b is

$$(1.6) \qquad U(a,b,z) = z^{1-b}U(1+a-b, 2-b,z).$$

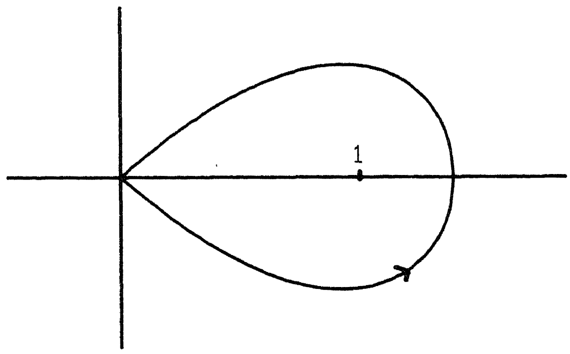

Figure 1, Countour for (1.5)

We consider in this note the asymptotic expansion of M and U for $a \to \infty$. These expansions will hold uniformly with respect to z in bounded domains that contain the point $z = 0$. Details about this uniformity aspect will be given further on. The parameter b will be kept fixed. The expansions contain modified Bessel functions.

Expansions of this nature are already available in the literature. The closest connection is SLATER (1960). In SKOVGAARD (1966) and OLVER (1974) also expansions of this type are considered. The author discussed such expansions (see TEMME (1975a)) when he gave a method for computing $U(k,1,z)$ for $k = 1,2,3,\ldots$.

The expansions given in the present paper are given with rigourous bounds for the remainders. The expansions we used in TEMME (1975a) lacked this property; also those in SLATER (1960) only have bounds in terms of order symbols. In Olver's book results are given from which strict error bounds can be derived. For practical use in numerical computations, however, these results are too complicated, whereas the bounds given in this paper are easily computed.

The expansions given here for U are used in a numerical algorithm for the computation of $U(a+k,b,z)$ for $k = 0,1,2,\ldots$, to be published in the near future. In that work the rigourous bounds play an essential role, especially for real values of a, b and z. Some features of the algorithm are announced in subsection 2.5.

The starting point for developing the confluent hypergeometric functions in terms of Bessel functions are the integrals (1.3) and (1.5). In the above mentioned references the starting point is Kummer's equation (1.1). In section 2 we give the results for the U-function, in section 3 for the M-function.

2. THE EXPANSION FOR U(a,b,z)

For large values of a and small values of $|z|$ (with Re $z > 0$) the main contributions in the integral in (1.3) come from the large t-values. By writing $t/(1+t) = \exp(-\tau)$ the integral becomes after some manipulations

$$U(a,b,z) = \frac{e^{z/2}}{\Gamma(a)} \int_0^\infty e^{-a\tau - z/\tau}\, \tau^{-b}\, f(\tau)\, d\tau, \tag{2.1}$$

where

$$\begin{aligned} f(\tau) &= e^{z\mu(\tau)} [\tau/(1-e^{-\tau})]^b \\ \mu(\tau) &= 1/\tau - 1/(e^{\tau}-1) - \tfrac{1}{2}. \end{aligned} \tag{2.2}$$

The function f is analytic in the strip $|\mathrm{Im}\ \tau| < 2\pi$ and it can be expanded for $|\tau| < 2\pi$ into

$$f(\tau) = \sum_{n=0}^\infty c_n(b,z)\tau^n, \tag{2.3}$$

where c_n are combinations of Bernoulli numbers and Bernoulli polynomials. In fact we have

$$\begin{aligned} \mu(t) &= -\sum_{n=1}^\infty \frac{B_{2n} t^{2n-1}}{(2n)!}, \\ [\tau/(1-e^{-\tau})]^b &= \sum_{n=0}^\infty \frac{B_n^{(b)}(b)}{n!}\, \tau^n. \end{aligned} \tag{2.4}$$

The expansion (2.3) is substituted in (2.1). For obtaining an expression for the remainder we write for N = 0,1,2,...

$$(2.5)\qquad f(\tau) = \sum_{n=0}^{N-1} c_n(b,z)\tau^n + \tau^N f_N(\tau),$$

and (2.1) becomes

$$(2.6)\qquad U(a,b,z) = \sum_{n=0}^{N-1} c_n(b,z)\phi_n(a,b,z) + R_N(a,b,z)$$

with

$$(2.7)\qquad \phi_n(a,b,z) = 2\,\frac{e^{z/2}}{\Gamma(a)}\,(z/a)^{(n+1-b)/2}\,K_{n+1-b}[2(az)^{\frac{1}{2}}],$$

$$(2.8)\qquad R_N(a,b,z) = \frac{e^{z/2}}{\Gamma(a)}\int_0^\infty e^{-z/\tau-a\tau}\,\tau^{N-b} f_N(\tau)\,d\tau.$$

In (2.7) $K_\nu(z)$ is the modified Bessel function, which can be represented as the integral

$$(2.9)\qquad K_\nu[2(z\zeta)^{\frac{1}{2}}] = \tfrac{1}{2}(\zeta/z)^{-\frac{1}{2}\nu}\int_0^\infty e^{-zt-\zeta/t}\,t^{-\nu-1}\,dt,$$

for Re $z > 0$, Re $\zeta > 0$, and which is an even function in ν. The many-valued functions appearing in (2.9) are supposed to be real for positive values of their arguments.

A bound for the remainder R_N can be constructed if we have a bound for f_N on $\tau \geq 0$. It is possible to continue with complex variables. However, the bounds obtained then are less realistic and some inequalities may loose their significance. Therefore we proceed with real values of a, b, and $z = x$ with the restriction

$$(2.10)\qquad x > 0,\quad a > 0,\quad b \in \mathbb{R}.$$

Negative values of b may be excluded by using (1.6).

Before giving the bound on R_N we give information how to interprete the expansion (2.6).

2.1. The asymptotic nature of the expansion

In this subsection we will prove that for $a \to \infty$

(i) $\{\phi_j(a,b,x)\}$ is an asymptotic sequence,

(ii) $R_N(a,b,x) = \mathcal{O}(\phi_N(a,b,x))$.

These properties are valid uniformly with respect to x in compact sets in $x \geq 0$ and uniformly with respect to b in compact subsets of $\mathbb{R}$; (ii) is valid if N is large enough, i.e., if $N > b$. The coefficients c_n of (2.6) do not depend on a and are polynomials in b and x. Therefore, if (i) is proved, we also have that $\{c_j\phi_j\}$ is an asymptotic sequence.

Before we prove (i) and (ii) we give some preliminary results.

LEMMA 1. *Let* $p(x) = K_\nu'(x)/K_\nu(x)$ *for* $x > 0$, $\nu \in \mathbb{R}$. *Then*

$$\frac{dp(x)}{dx} \geq 0, \qquad x > 0.$$

PROOF. This follows from the known fact that the modified Bessel function is log-convex. A direct proof is as follows. Take the well-known integral

$$K_\nu(x) = \int_0^\infty e^{-x\cosh t} \cosh \nu t \, dt.$$

Then by using Cauchy-Schwartz' inequality it easily follows that $[K_\nu'(x)]^2 \leq$ $\leq K_\nu(x)K_\nu''(x)$, which has to be proved. □

LEMMA 2. *For* $x > 0$, $\nu \in \mathbb{R}$ *we have*

$$K_{\nu+1}(x)/K_\nu(x) \leq \frac{2\nu+1+[1+4(x^2+\nu^2)]^{\frac{1}{2}}}{2x}.$$

PROOF. From the differential equation

$$x^2K_\nu''(x) + xK_\nu'(x) - (x^2+\nu^2)K_\nu(x) = 0$$

it follows that the function p introduced in Lemma 1 satisfies the equation

$$p^2 + p/x = (x^2+\nu^2)/x^2 - p'.$$

From Lemma 1 we have $p' \geq 0$ for $x > 0$, hence

$$p^2 + p/x \leq (x^2+\nu^2)/x^2, \qquad x > 0,$$

from which follows that (we know that $p(x) < 0$)

$$-\frac{1+[1+4(x^2+\nu^2)]^{\frac{1}{2}}}{2x} \leq p(x) < 0.$$

From the recurrence relation for $K_\nu(x)$

$$K_{\nu+1}(x) = \frac{\nu}{x} K_\nu(x) - K_\nu'(x)$$

we obtain $K_{\nu+1}(x)/K_\nu(x) = \frac{\nu}{x} - p(x)$. Using the above inequality for p, we conclude that the lemma is proved. □

THEOREM 1. *For* $x > 0$, $a > 0$, $b \in \mathbb{R}$, $j = 1,2,3,\ldots$ *we have*

$$\text{(2.11)} \qquad \frac{\phi_j(a,b,x)}{\phi_{j-1}(a,b,x)} \leq \frac{2(j-b)+1+[1+4(j-b)^2+16xa]^{\frac{1}{2}}}{4a}.$$

PROOF. This follows immediately from Lemma 2 and (2.7). □

COROLLARY 1. $\{\phi_j(a,b,x)\}$ *is an asymptotic sequence for* $a \to \infty$, *uniformly in bounded* b-*intervals and bounded* x-*intervals* $(x \geq 0)$.

The next thing to prove is property (ii) (see the beginning of this subsection). It will turn out that we have to restrict the b-values. In fact we have

THEOREM 2. *For* $a \to \infty$, $n = 0,1,2,\ldots$, $b \in \mathbb{R}$ *such that* $n > b$ *we have*

$$\text{(2.12)} \qquad R_n(a,b,x) = \mathcal{O}(\phi_n(a,b,x))$$

uniformly in compact subsets in $x \geq 0$.

PROOF. From Lemma 5 (to be proved in the next subsection) it follows that f_n appearing in (2.8) satisfies

(2.13) $|f_n(\tau)| \le M_n(b,x)$,

$n = 0,1,2,\ldots$, $n > b$. $M_n(b,x)$ are assignable constants, which will be computed in §2.2 for the case that $b \ge 0$ and $n \ge 2+b$. Hence, if we use these bounds we have from (2.8)

(2.14) $R_n(a,b,x) \le M_n(b,x)\phi_n(a,b,x)$,

from which the theorem follows. □

From Theorems 1 and 2 the asymptotic nature of (2.6) is established. It is possible to refine the above results, for instance by constructing sharper bounds for the remainder and by enlarging the domain of uniformity with respect to x (in OLVER (1975) and SKOVGAARD (1966) expansions are given holding in unbounded x-domains). However, our scope is to give expansions which can easily be used for instance for numerical computations. From the construction of the bound of f_n to be given in §2.2 and from (2.6) and (2.14) it appears that our result indeed meet this condition. Of course, an algorithm for the Bessel function $K_\nu(x)$, $\nu \in \mathbb{R}$, $x > 0$ must be available, but this was settled in TEMME (1975b).

2.2. Construction of the bound for f_n

The function f of (2.2) has singularities in the τ-values $\pm 2\pi i$, $\pm 4\pi i, \ldots$. Those in $\pm 2\pi i$ are nearest to the real τ-line and they have a main influence upon the behaviour of the coefficients c_n and the function f_n in (2.5). From Taylor's theorem we know that f_n can be written as

(2.15) $$f_n(\tau) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-\tau)w^n}\, dw$$

where C is a contour as drawn in Figure 2. The points $w = 0$ and $w = \tau$ are inside C, whereas the above mentioned singularities of f are outside C. This countour will be deformed into two straight lines

(2.16) $C_d = \{w \mid w \in \mathbb{C},\ \mathrm{Im}\, w = \pm d,\ \ 0 < d < 2\pi\}$,

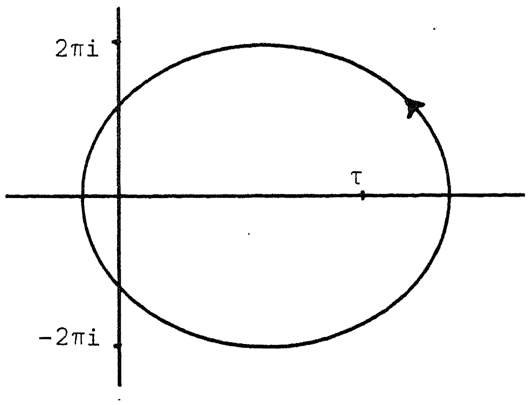

Figure 2, Contour for (2.15)

where it is convenient, as will appear, to have $3\pi/2 \le d < 2\pi$. This choic of $C = C_d$ is possible if $n > b$. Next we compute a bound for f on C_d. Let ı write (see (2.2))

$$(2.17)\qquad \begin{aligned} f(\tau) &= f_1(\tau,x)f_2(\tau,b) \\ f_1(\tau,x) &= \exp[x\mu(\tau)], \quad f_2(\tau,b) = [\tau/(1-e^{-\tau})]^b. \end{aligned}$$

We then have

<u>LEMMA 3</u>. *Let* $b \ge 0$, $w \in C_d$, $w = u \pm id$, $3\pi/2 \le d < 2\pi$. *Then*

$$(2.18)\qquad f_2(w,b) \le |w/\sin d|^b.$$

<u>PROOF</u>. $|1/(1-e^{-w})| = (1-2\cos d\, e^{-u}+e^{-2u})^{-\frac{1}{2}}$. The maximum of this function (as a function of real u) occurs at the point u_0 satisfying $e^{-u_0} = \cos d$, and this maximum is $1/|\sin d|$. □

<u>LEMMA 4</u>. *Let* $x > 0$, $w \in C_d$, $w = u \pm id$, $3\pi/2 \le d < 2\pi$. *Then*

$$(2.19)\qquad f_1(w,x) \le \exp[\tfrac{1}{2}x(1/d+1/|\sin d|)].$$

<u>PROOF</u>. We write $|f_1(w,x)| = \exp[x(h_1(u,d) + h_2(u,d))]$,

with

$$h_1(u,d) = u/(u^2+d^2), \quad h_2(u,d) = -\tfrac{1}{2}\,\frac{\sinh u}{\cosh u - \cos d}\,.$$

By computing the maximal values of h_1 and h_2 (as functions of u), the result is easily verified. □

Combining the two Lemmas we have for $x > 0$, $b \geq 0$, $w \in C_d$, $w = u \pm id$, $3\pi/2 \leq d < 2\pi$

(2.20) $$|f(w)| \leq |w|^b\, K(d,b,x)$$

with

(2.21) $$K(d,b,x) = |\sin d|^{-b} \exp[\tfrac{1}{2}x(1/d+1/|\sin d|)].$$

From this follows the result we used in (2.13)

LEMMA 5. *For* $\tau \geq 0$, $n \geq 2 + b$, $x > 0$, $b \geq 0$, *we have*

(2.22) $$|f_n(\tau)| \leq d^{b-n}\, K(d,b,x).$$

PROOF. From the above estimate for f it follows that

$$|f_n(\tau)| \leq \pi^{-1}\, K(d,b,x) \int_{-\infty}^{\infty} |u+id|^{b-n} |u+id-\tau|^{-1} du.$$

Since $|u+id-\tau| \geq d$, we have

$$|f_n(\tau)| \leq 2d^{-1}\pi^{-1}\, K(d,b,x) \int_0^{\infty} (u^2+d^2)^{(b-n)/2}\, du$$

$$= 2\pi^{-1}\, d^{b-n}\, K(d,b,x) \int_0^{\infty} (t^2+1)^{(b-n)/2} dt.$$

The integral converges if $b-n < -1$, or $n > 1+b$. We obtain a simple bound if we take $(b-n)/2 \leq -1$, or $n \geq 2+b$ and replacing $(b-n)/2$ in the integrand by -1. The result is thus established. □

REMARK. Lemma 5 gives the bound in (2.13) by taking

(2.23) $$M_n(b,x) = d^{b-n} K(d,b,x).$$

This is only possible if $n \geq 2+b$ and $b \geq 0$. It is easily seen however that, as stated in the proof of Theorem 2, for $n > b$ also constants $M_n(b,x)$ exist satisfying (2.13). For $n \geq 2+b$, $b \geq 0$, we constructed these numbers.

2.3. About the choice of d

The free parameter d will now be chosen in order to make the numbers $M_n(b,x)$ as small as possible. The above analysis can be done for all $d \in (0,2\pi)$, although $K(d,b,x)$ should be modified then. The point is to choose d as large as possible in order to make d^{b-n} in (2.23) as small as possible. But on the other hand $K(d,b,x)$ is not bounded as $d \to 2\pi$. If we fix x and b, and if we also fix the number of terms (n) in the expansion (2.6) (with $n \geq 2+b$), we can try to choose an optimal value of d in $(3\pi/2,2\pi)$. The word "optimal" should be understood in an approximative way. For instance, the factor $|\sin d|^{-b}$ in $K(d,b,x)$ of (2.21) will be omitted in the determination of d.

If $d \sim 2\pi$, we have $|\sin d| \sim 2\pi - d$, $d < 2\pi$. We compute d such that it makes

$$d^{b-n} \exp[\tfrac{1}{2}x/(2\pi-d)]$$

as small as possible on $(0,2\pi)$. This value follows from the equation $\tfrac{1}{2}x/(2\pi-d)^2 = (n-b)/d$, from which we obtain (if we want to have $3\pi/2 \leq d < 2\pi$)

(2.24) $$d = \max(3\pi/2,\ 2\pi + \tfrac{1}{4}x/(n-b) - \tfrac{1}{2}[\tfrac{1}{4}x^2/(n-b)^2 + 4\pi x/(n-b)]^{\frac{1}{2}}).$$

Summarizing the above results we conclude that the remainder R_N in (2.6) satisfies for $N \geq 2+b$, $b \geq 0$ the inequality

(2.25) $$|R_N(a,b,x)| \leq d^{b-N} K(d,b,x)\phi_N(a,b,x)$$

with K given in (2.21) and all d in $(3\pi/2,2\pi)$; a suitable choice for d is given by (2.24). For negative values of b we can use (1.6).

2.4. An expansion for U'(a,b,x)

Differentiating (2.1) with respect to z gives

$$(2.26)\qquad U'(a,b,x) = -\frac{e^{\frac{1}{2}x}}{\Gamma(a)}\int_0^\infty e^{-a\tau-x/\tau}\,\tau^{-b-1}\,g(\tau)\,d\tau$$

with

$$(2.27)\qquad g(\tau) = \frac{\tau}{e^{\tau}-1}\,f(\tau) = e^{-\tau}\,\frac{\tau}{1-e^{-\tau}}\,f(\tau).$$

Hence, from (2.2) and (2.20) we obtain that for $w \in C_d$, $w = u \pm id$, $x > 0$, $3\pi/2 \le d < 2\pi$, $b \in \mathbb{R}$

$$(2.28)\qquad |g(w)| \le |w|^{b+1}\,K(d,b+1,x)$$

and that $g_n(\tau)$ defined by

$$(2.29)\qquad g(\tau) = \sum_{m=0}^{n-1} d_m(b,x)\tau^m + \tau^n\,g_n(\tau)$$

satisfies for $\tau \ge 0$, $3\pi/2 \le d < 2\pi$, $n \ge b+3$, $b+1 \ge 0$,

$$(2.30)\qquad |g_n(\tau)| \le d^{b+1-n}\,K(d,b+1,x).$$

By expanding we obtain for $a > 0$, $x > 0$, $b \ge -1$, $n = 0,1,2,\ldots$

$$(2.31)\qquad U'(a,b,x) = -\sum_{n=0}^{N-1} d_n(b,x)\phi_n(a,b+1,x) + T_N(a,b,x)$$

with (for $N \ge 3+b$)

$$(2.32)\qquad |T_N(a,b,x)| \le d^{b+1-N}\,K(d,b+1,x)\phi_N(a,b+1,x).$$

Remark that $\phi_n(a,b+1,x) = \phi_{n-1}(a,b,x)$. Hence for obtaining U'(a,b,x) we can use the same functions ϕ_j as for U(a,b,x). Moreover, the coefficients d_n in (2.29) and (2.31) are closely related with c_n in (2.5) and (2.6).

This follows from well-known properties of the Bernoulli polynomials appearing in (2.4). To show this we remember that

$$(2.33)\qquad e^{x\tau}[\tau/(e^{\tau}-1)]^m = \sum_{n=0}^{\infty} \frac{B_n^{(m)}(x)}{n!}\tau^n,$$

giving the second relation in (2.4). For g(t) of (2.27) we need the coefficients $\{a_j\}$ in

$$(2.34)\qquad \frac{\tau}{e^{\tau}-1}[\tau/(1-e^{-\tau})]^b = \sum_{n=0}^{\infty} a_n\tau^n.$$

According to (2.33) we have $a_n = B_n^{(b+1)}(b)/n!$.

Upon differentiating both sides of the second of (2.4) with respect to τ, we obtain

$$(2.35)\qquad bB_n^{(b+1)}(b) = (b-n)B_n^{(b)}(b).$$

Hence the a_n in (2.34) satisfy

$$(2.36)\qquad a_n = (1-n/b)B_n^{(b)}(b)/n!.$$

From this follows that if the coefficients appearing in the second of (2.4) are available (which are needed for the computation of c_n in (2.6)), then the coefficients a_n in (2.34) are available too (these are needed for the computation of d_n in (2.31)).

2.5. The use of the results for numerical computations

We use the above results in an algorithm for computing the values

$$U(a+k,b,x),\qquad k = 0,1,\ldots,K,$$

for $0 < a \le 1$, $x > 0$, $b \in \mathbb{R}$. When x is not small, say $x \ge 1$ we compute these values by using an recursion relation for the U-function and Miller's algorithm. If x is in the interval (0,1) the convergence of this algorithm becomes poorer, according as x becomes smaller. The idea is then to choose an integer m, $m \geq K$, and to compute $U(a+m,b,x)$, $U'(a+m,b,x)$ by using the expansions given in (2.6) and (2.31). In these series we fix the number of terms (say $N = 10$) and then we choose $m \geq K$ such that the remainders in the expansions fall below the desired accuracy. Details of this process will be given in a future publication. The restriction on the parameter b in (2.25) and (2.32) is not very important. If b is larger than n-2 or n-3 then we can use recursion with respect to the b-parameter. If $b < 0$, we can use (1.6).

3. THE EXPANSION FOR M(a,b,z)

The starting point is the integral (1.5) with as contour the circle $|t-1| = 1$. The transformation $t = \tau/(\tau-1)$ transforms this circle into itself. To see this we write $\tau = t/(t-1)$. With $t = 1 + \exp(i\phi)$, $0 \leq \phi \leq 2\pi$, we obtain $\tau = 1 + \exp(-i\phi)$. The result is

$$M(a,b,z) = \frac{\Gamma(1+a-b)\Gamma(b)}{\Gamma(a)2\pi i} \int_C \tau^{a-1}(\tau-1)^{-b}\exp[z\tau/(\tau-1)]d\tau.$$

Next we take $\tau = e^w$; with $\tau = 1 + \exp(i\theta)$, $-\pi < \theta \leq \pi$, we see that the circle C in the w-plane is described by $w = u + iv$, $u = \ln(2\cos v)$, $-\frac{1}{2}\pi < v \leq \frac{1}{2}\pi$. After some manipulations we obtain

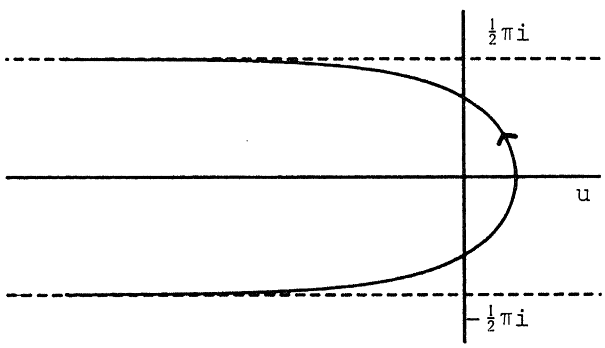

Figure 3, Contour for (3.1)

$$(3.1)\qquad M(a,b,z) = \frac{\Gamma(1+a-b)\Gamma(b)e^{\frac{1}{2}z}}{\Gamma(a)2\pi i}\int_L f(-w)e^{z/w+aw}w^{-b}dw$$

where f is defined in (2.2). Upon substituting the expansion (2.5), we obtain

$$(3.2)\qquad M(a,b,z) = \sum_{n=0}^{N-1} (-1)^n c_n(b,z)\psi_n(a,b,z) + S_N(a,b,z),$$

with

$$(3.3)\qquad \psi_n(a,b,z) = \frac{\Gamma(1+a-b)\Gamma(b)}{\Gamma(a)} e^{\frac{1}{2}z} (z/a)^{(n+1-b)/2} I_{b-n-1}[2(za)^{\frac{1}{2}}],$$

where we used the well-known integral representation of the modified Bessel function $I_\nu(z)$, namely Schläfli's integral

$$(3.4)\qquad I_\nu(z) = \frac{1}{2\pi i}\int_L u^{-\nu-1} e^{\frac{1}{2}z(u+1/u)}du$$

where L can be chosen as in Figure 3. For the remainder S_N we have the expression

$$(3.5)\qquad S_N(a,b,z) = \frac{\Gamma(1+a-b)\Gamma(b)}{\Gamma(a)2\pi i} e^{\frac{1}{2}z}(-1)^N \int_L f_N(-w)w^{N-b} e^{z/w+aw} dw.$$

In the above formulas (3.1), (3.2), (3.3) and (3.5) we can give z any finite complex value, a should satisfy Re a > 0, and 1 + a - b $\neq$ 0,-1,-2,... . For b = 0, -1,-2,... M(a,b,z), $\psi_n(a,b,z)$ and $S_n(a,b,z)$ have simple poles. For applications it is necessary to exclude small neighbourhoods of these b-values.

We may proceed the analysis as in section 2. That involves investigations on the asymptotic nature of (3.2), inequalities for $\psi_j(a,b,z)/\psi_{j-1}(a,b,z)$ and bounds for $S_n(a,b,z)$. The present case, however, is not as nice as the foregoing. This is partly due to the integration over complex w-values in (3.5), whereas (2.8) is an integral over real t-values. Another point is the behaviour of the modified Bessel functions appearing in (3.3). For deriving (2.11) some properties of the K_ν-function were used; it did not matter whether the order of the Bessel functions in Lemma 1 and 2 was positive or negative. But in the present case we must take into account negative orders for I_ν; for negative ν this function is not so easy to handle. For instance, $I_\nu(x)$ is not monotone for negative ν-values and it has zeros there.

Finally I remark that the analysis for the U-function was motivated by my work on constructing algorithms for this function. For the M-function I am not so interested in these aspects.

We finish this section with an indication how to obtain a bound for S_N in (3.5), for real $a > 0$, $z = x > 0$. Take for L the contour

$$L = \{w \mid |w| = (x/a)^{\frac{1}{2}}\} \cup \{w \mid \mathrm{Re}\, w \le (x/a)^{\frac{1}{2}},\ \mathrm{Im}\, w = 0\},$$

where we suppose that x and a are such that $(x/a)^{\frac{1}{2}} < 2\pi$. Suppose furthermore that we have bounds $M_n(b,x)$ for $|f_n|$ on this L. Then

$$|S_n(a,b,x)| \le \frac{|(b)\Gamma(1+a-b)|}{\Gamma(a)2\pi} e^{\frac{1}{2}z} M_n(b,x)$$

$$\left[\int_{\sqrt{x/a}}^{\infty} w^{n-b} e^{-x/w-aw} dw + (x/a)^{(n-b+1)/2} \int_0^{2\pi} e^{2\sqrt{xa}\cos\phi} d\phi\right].$$

The first integral can be estimated by a K_ν-function (see (2.9)), the second one equals $2\pi I_0(2\sqrt{xa})$.

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